Polynomial semantics of probabilistic circuits

Oliver Broadrick, Honghua Zhang, and Guy Van den Broeck

University of California, Los Angeles
## Probabilistic Models

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>.4</td>
</tr>
</tbody>
</table>
Probabilistic Models

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>.4</td>
</tr>
</tbody>
</table>

Transformers
Diffusion models
VAEs

Expressive-efficient
## Probabilistic Models

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>.4</td>
</tr>
</tbody>
</table>

- Fully factorized
  - HMMs
  - Mixture models

- Transformers
  - Diffusion models
  - VAEs

- Expressive-efficient
- Tractable
Probabilistic Models

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>.4</td>
</tr>
</tbody>
</table>

Tractable

Expressive-efficient

Fully factorized
HMMs
Mixture models

Transformers
Diffusion models
VAEs
**Marginal Inference**

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>.4</td>
</tr>
</tbody>
</table>

\[
\Pr[X_1 = 1] = \Pr[X_1 = 1, X_2 = 0] + \Pr[X_1 = 1, X_2 = 1] = 0.3 + 0.4 = 0.7
\]
Marginal Inference

\[
\begin{array}{c|cc|c}
X_1 & X_2 & Pr \\
0 & 0 & .1 \\
0 & 1 & .2 \\
1 & 0 & .3 \\
1 & 1 & .4 \\
\end{array}
\]

\[
\Pr[X_1 = 1] = \Pr[X_1 = 1, X_2 = 0] + \Pr[X_1 = 1, X_2 = 1] \\
= 0.3 + 0.4 \\
= 0.7
\]

**Goal:** Find maximally expressive-efficient models that support marginal inference in time polynomial in the model size.
Approaches

Bayesian Networks (of bounded treewidth)
Determinantal Point Processes
Characteristic Circuits
Multi-Linear Representations
Probabilistic Generating Circuits
Sum-Product Networks

...
Approaches

Bayesian Networks (of bounded treewidth)
Determinantal Point Processes
Characteristic Circuits
Multi-Linear Representations
Probabilistic Generating Circuits
Sum-Product Networks

...
Approaches

- Bayesian Networks (of bounded treewidth)
- Determinantal Point Processes
- Characteristic Circuits
- Multi-Linear Representations
- Probabilistic Generating Circuits
- Sum-Product Networks

- Likelihood polynomial
- Network polynomial
- Generating polynomial
- Fourier polynomial
Circuits represent polynomials succinctly

\[ 3x_1x_2 + x_1x_3 + 6x_2^2 + 2x_2x_3 \]
Circuits represent polynomials succinctly

\[ 3x_1x_2 + x_1x_3 + 6x_2^2 + 2x_2x_3 \]

Circuits are fully expressive
Circuits represent polynomials succinctly

\[ 3x_1x_2 + x_1x_3 + 6x_2^2 + 2x_2x_3 \]

Circuits are *fully expressive*

They can also be *expressive-efficient*
Polynomial Semantics

- Network polynomial
- Generating polynomial
- Likelihood polynomial
- Fourier polynomial

Darwiche [2003]
Roth and Samdani [2009]
Yu et al. [2023]
Zhang et al. [2021]
Polynomial Semantics

Darwiche [2003]
Network polynomial

Roth and Samdani [2009]
Likelihood polynomial

Zhang et al. [2021]
Generating polynomial

Yu et al. [2023]
Fourier polynomial
Polynomial Semantics

- Darwiche [2003]: Network polynomial
- Roth and Samdani [2009]: Likelihood polynomial
- Zhang et al. [2021]: Generating polynomial
- Yu et al. [2023]: Fourier polynomial
Polynomial Semantics

- Network polynomial
  - Darwiche [2003]
  - Roth and Samdani [2009]
- Generating polynomial
  - Zhang et al. [2021]
  - Yu et al. [2023]
- Likelihood polynomial
- Fourier polynomial
Polynomial Semantics

- Network polynomial
- Generating polynomial
- Likelihood polynomial
- Fourier polynomial

Network polynomial

\[ p(x_1, x_2, \bar{x}_1, \bar{x}_2) = 0.1\bar{x}_1\bar{x}_2 + 0.2\bar{x}_1x_2 + 0.3x_1\bar{x}_2 + 0.4x_1x_2 \]

<table>
<thead>
<tr>
<th>(X_1)</th>
<th>(X_2)</th>
<th>(\text{Pr})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.4</td>
</tr>
</tbody>
</table>
Network polynomial

\[ p(x_1, x_2, \bar{x}_1, \bar{x}_2) = \cdot1\bar{x}_1\bar{x}_2 + \cdot2\bar{x}_1x_2 + \cdot3x_1\bar{x}_2 + \cdot4x_1x_2 \]

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>.4</td>
</tr>
</tbody>
</table>
Network polynomial

\[ p(x_1, x_2, \bar{x}_1, \bar{x}_2) = .1\bar{x}_1\bar{x}_2 + .2\bar{x}_1x_2 + .3x_1\bar{x}_2 + .4x_1x_2 \]

<table>
<thead>
<tr>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( \text{Pr} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>.4</td>
</tr>
</tbody>
</table>

\( \text{Pr}[X_1 = 1] \)
Network polynomial

\[ p(x_1, x_2, \bar{x}_1, \bar{x}_2) = .1\bar{x}_1\bar{x}_2 + .2\bar{x}_1x_2 + .3x_1\bar{x}_2 + .4x_1x_2 \]

<table>
<thead>
<tr>
<th>(X_1)</th>
<th>(X_2)</th>
<th>(Pr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>.4</td>
</tr>
</tbody>
</table>

\[ \Pr[X_1 = 1] = p(1, 1, 0, 1) \]
\[ = .1(0)(1) + .2(0)(1) + .3(1)(1) + .4(1)(1) \]
\[ = 0 + 0 + .3 + .4 \]
\[ = .7 \]
Network polynomial

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>.4</td>
</tr>
</tbody>
</table>

\[ X_1 X_2 \Pr^{0.1} 0.2 0.4 \]
\[ \bar{x}_2 \times x_2 \times x_1 \]
\[ x_1 \bar{x}_1 \]
Network polynomial

$$X_1 X_2 \Pr$$

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$\Pr$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.4</td>
</tr>
</tbody>
</table>

$$\Pr[X_1 = 1]?$$
Network polynomial

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>.4</td>
</tr>
</tbody>
</table>

$\Pr[X_1 = 1]?$
Network polynomial

\[ \text{Pr}[X_1 = 1]? \]

<table>
<thead>
<tr>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( \text{Pr} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.4</td>
</tr>
</tbody>
</table>

\[ x_1 \times \bar{x}_1 + x_2 \times 0.75 + \bar{x}_2 \times 1 = 0.7 \]
Progress Update

Network polynomial

Generating polynomial

Likelihood polynomial

Fourier polynomial
Progress Update

Network polynomial

Generating polynomial

Likelihood polynomial

Fourier polynomial
Likelihood polynomial

\[ p(x_1, x_2) = 0.2x_1 + 0.1x_2 + 0.1 \]

<table>
<thead>
<tr>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( \text{Pr} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.4</td>
</tr>
</tbody>
</table>
Likelihood polynomial

\[ p(x_1, x_2) = .2x_1 + .1x_2 + .1 \]

Marginal inference?

<table>
<thead>
<tr>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( \text{Pr} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>.4</td>
</tr>
</tbody>
</table>
Likelihood polynomial

\[ p(x_1, x_2) = .2x_1 + .1x_2 + .1 \]

Marginal inference?
Relation to network polynomial?

<table>
<thead>
<tr>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>.4</td>
</tr>
</tbody>
</table>
Likelihood polynomial

\[ p(x_1, x_2) = .2x_1 + .1x_2 + .1 \]

Marginal inference?
Relation to network polynomial?

<table>
<thead>
<tr>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>.4</td>
</tr>
</tbody>
</table>

Transform network to likelihood:

\[ p(x, \bar{x}) = .1\bar{x}_1\bar{x}_2 + .2\bar{x}_1x_2 + .3x_1\bar{x}_2 + .4x_1x_2 \]

- Replace \( \bar{x}_i \) with \( 1 - x_i \)
Likelihood polynomial

Transform likelihood to network:

\[ p(x_1, x_2) = 0.2x_1 + 0.1x_2 + 1 \]
Likelihood polynomial

Transform likelihood to network:

\( p(x_1, x_2) = .2x_1 + .1x_2 + .1 \)

\[
(x_1 + \bar{x}_1)(x_2 + \bar{x}_2) \left( .2 \frac{x_1}{x_1 + \bar{x}_1} + .1 \frac{x_2}{x_2 + \bar{x}_2} + .1 \right)
\]
Transform likelihood to network:

\[ p(x_1, x_2) = 0.2x_1 + 0.1x_2 + 1 \]

\[ = \frac{0.2}{x_1 + \bar{x}_1} + \frac{0.1}{x_2 + \bar{x}_2} + 0.1 \]

\[ = \frac{0.2}{x_1 + \bar{x}_1} x_2 + \frac{0.1}{x_2 + \bar{x}_2} x_1 + \frac{0.1}{x_1 + \bar{x}_1} x_2 + \frac{0.1}{x_2 + \bar{x}_2} x_1 + 0.1 \]
Likelihood polynomial

Transform likelihood to network:

\[ p(x_1, x_2) = 0.2x_1 + 0.1x_2 + 1 \]

\[ (x_1 + \bar{x}_1)(x_2 + \bar{x}_2) \left( 0.2 \frac{x_1}{x_1 + \bar{x}_1} + 0.1 \frac{x_2}{x_2 + \bar{x}_2} + 1 \right) \]

\[ = 0.2x_1(x_2 + \bar{x}_2) + 0.1x_2(x_1 + \bar{x}_1) + 0.1(x_1 + \bar{x}_1)(x_2 + \bar{x}_2) \]

\[ = p(x_1, x_2, \bar{x}_1, \bar{x}_2) \]
Transform likelihood to network:
Transform likelihood to network:

\[ \text{Likelihood polynomial} \]

\[ \text{Likelihood} \]

\[ x_1 \cdot \ldots \cdot x_n \]

\[ \prod_i (x_i + \bar{x}_i) \]

\[ \frac{x_1}{x_1 + \bar{x}_1} \cdot \ldots \cdot \frac{x_n}{x_n + \bar{x}_n} \]
Removing Divisions

Theorem (Strassen [1973]).  You can remove divisions in polynomial time!
Removing Divisions

Theorem (Strassen [1973]). You can remove divisions in polynomial time!
Removing Divisions

Theorem (Strassen [1973]). You can remove divisions in polynomial time!
Likelihood polynomial

Transform likelihood to network:

\[
\prod_i (x_i + \bar{x}_i)
\]
Transform likelihood to network:

\[ \text{Likelihood to Network:} \]

\[ L(x_1, \ldots, x_n) = \prod_{i} (x_i + \bar{x}_i) \]

\[ \frac{x_1}{x_1 + \bar{x}_1} \cdots \frac{x_n}{x_n + \bar{x}_n} \]
Progress Update

Network polynomial

Generating polynomial

Likelihood polynomial

Fourier polynomial
Progress Update

- Network polynomial
- Generating polynomial
- Likelihood polynomial
- Fourier polynomial
Progress Update

Network polynomial

Generating polynomial

Likelihood polynomial

Fourier polynomial
Generating polynomial
Generating polynomial

Monotone, decomposable circuits computing network polynomials (SPNs, PCs)
Generating polynomial

Monotone, decomposable circuits computing network polynomials (SPNs, PCs)

Circuits computing generating polynomials
Monotone, decomposable circuits computing network polynomials (SPNs, PCs)

Circuits computing generating polynomials

Spanning tree distribution

Martens and Medabalimi [2015], Zhang et al. [2021]
Generating polynomial

$$g(x) = .1 + .2x_2 + .3x_1 + .4x_1x_2$$

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>.4</td>
</tr>
</tbody>
</table>
Generating polynomial

\[ g(x) = .1 + .2x_2 + .3x_1 + .4x_1x_2 \]

Marginal inference: ✔️ [Zhang et al., 2021]
Generating polynomial

\[ g(x) = .1 + .2x_2 + .3x_1 + .4x_1x_2 \]

Marginal inference: ✓ [Zhang et al., 2021]
Relation to network polynomial?

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>.1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>.2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>.4</td>
</tr>
</tbody>
</table>
Generating polynomial

\[ g(x) = .1 + .2x_2 + .3x_1 + .4x_1x_2 \]

Marginal inference: [Zhang et al., 2021]
Relation to network polynomial?

Transform network to generating:
\[ p(x_1, x_2, \bar{x}_1, \bar{x}_2) = .1\bar{x}_1\bar{x}_2 + .2\bar{x}_1x_2 + .3x_1\bar{x}_2 + .4x_1x_2 \]

- Replace \( \bar{x}_i \) with 1
Generating polynomial

Transform generating to network:

Generating

\[ x_1 \quad \ldots \quad x_n \]

Generating

\[ \frac{x_1}{\bar{x}_1} \quad \ldots \quad \frac{x_n}{\bar{x}_n} \]

Network

\[ x_1 \bar{x}_1 \quad \ldots \quad \bar{x}_n x_n \]
Progress Update

Network polynomial

Generating polynomial

Likelihood polynomial

Fourier polynomial
Progress Update

Network polynomial

Generating polynomial

Likelihood polynomial

Fourier polynomial
Progress Update

Network polynomial

Generating polynomial

Likelihood polynomial

Fourier polynomial
Fourier Polynomial

Fourier transform of the probability mass function

Graphical model approximate inference

• Characteristic Circuits

Proposition. Generating polynomials and Fourier polynomials compute the same function on respective domains \([-1, 1]^n\) and \([0, 1]^n\).
Fourier Polynomial

Fourier transform of the probability mass function
Fourier Polynomial

Fourier transform of the probability mass function

- Graphical model approximate inference
- Characteristic Circuits
Proposition. Generating polynomials and Fourier polynomials compute the same function on respective domains $\{-1, 1\}^n$ and $\{0, 1\}^n$. 

Fourier Polynomial 

Fourier transform of the probability mass function 

- Graphical model approximate inference 
- Characteristic Circuits
Progress Update

Network polynomial

Generating polynomial

Likelihood polynomial

Fourier polynomial
Progress Update

Network polynomial

Generating polynomial

Likelihood polynomial

Fourier polynomial
Some New Semantics

Network polynomial

Generating polynomial

Likelihood polynomial

Fourier polynomial
Non-binary variables?

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>.1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>.3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>.2</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
</tbody>
</table>
Non-binary variables?

Literature: just use a binary encoding

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>.1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>.3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Generating polynomial

Theorem. For $|K| \geq 4$, computing likelihoods on a circuit for $g(x)$ is $\#P$-hard.
Non-binary variables?

Literature: just use a binary encoding

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>.1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>.3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>.2</td>
</tr>
</tbody>
</table>

$g(x) = .1x_2 + .3x_1x_2^3 + .2x_1^3x_2^2 + \ldots$

Generating polynomial
Non-binary variables?

Literature: just use a binary encoding

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>.1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>.3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>.2</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

$$g(x) = \cdot 1x_2 + \cdot 3x_1x_2^3 + \cdot 2x_1^3x_2^2 + \ldots$$

Theorem. *For $|K| \geq 4$, computing likelihoods on a circuit for $g(x)$ is #$P$-hard.*

Approach: Reduction from 0, 1-permanent.
Conclusion

What we’ve done:

- Shown several distinct circuit models are equally expressive-efficient
- Unified existing (and one new) inference algorithms
- Inference is \#P-hard in generating polynomials circuits for \( k \geq 4 \) categories
Conclusion

What we’ve done:

- Shown several distinct circuit models are equally expressive-efficient
- Unified existing (and one new) inference algorithms
- Inference is $\#P$-hard in generating polynomials circuits for $k \geq 4$ categories

What’s next?

- How can this theoretical progress be leveraged in practice?
- Are there more expressive-efficient tractable representations?
Conclusion

What we’ve done:

- Shown several distinct circuit models are equally expressive-efficient
- Unified existing (and one new) inference algorithms
- Inference is \#P-hard in generating polynomials circuits for \( k \geq 4 \) categories

What’s next?

- How can this theoretical progress be leveraged in practice?
- Are there more expressive-efficient tractable representations?

Thank you! Questions?